A note about conditional Ornstein-Uhlenbeck processes

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11 janvier 2008

Abstract

In this short note, the identity in law, which was obtained by P. Salminen [6], between on one hand, the Ornstein-Uhlenbeck process with parameter γ, killed when it reaches 0, and on the other hand, the 3-dimensional radial Ornstein-Uhlenbeck process killed exponentially at rate γ and conditioned to hit 0, is derived from a simple absolute continuity relationship.

Keywords: Ornstein-Uhlenbeck process, Doob's h-transform, absolute continuity relationship.
Mathematical Subject Classification: 60J60; 60G15.
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- All probability distributions considered in this note are defined on $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, where $(X_t, t \geq 0)$ denotes the coordinate process, and $\mathcal{F}_t = \sigma\{X_s, s \leq t\}$, its natural filtration.

- For $\gamma > 0$, and $a > 0$, we denote by $\mathbb{P}_a^\gamma$ the law of the Ornstein-Uhlenbeck process with parameter γ, starting from a, i.e : under $\mathbb{P}_a^\gamma$, one has :
$$X_t = a + B_t - \gamma \int_0^t X_s \, ds \tag{1}$$
for a Brownian motion $(B_t, t \geq 0)$, starting from 0.

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Note that $(X_t \exp(\gamma t), t \geq 0)$ is a $\mathbb{P}_a^\gamma$ martingale which is indeed equal to :

$$a + \int_0^t \exp(\gamma s)\, dB_s,\ t \geq 0, \tag{2}$$

thus leading us to Doob's well-known representation of the Ornstein-Uhlenbeck process (see [2]) :

$$X_t = e^{-\gamma t}\Big(a + \beta(\frac{e^{2\gamma t} - 1}{2\gamma})\Big),\ t \geq 0, \tag{3}$$

for another Brownian motion $(\beta(u), u \geq 0)$, starting from 0.

- In this note, the 3-dimensional Orstein-Uhlenbeck process $(\vec{X}_t)$ with parameter γ and starting from $\vec{a}$, i.e : a solution of (1) where (B_t) is replaced by a 3-dimensional Brownian motion $(\vec{B}_t)$, and its radial part $R_t = |\vec{X}_t|$ play an important role; (R_t) solves the SDE :

$$R_t = a + \tilde{B}_t + \int_0^t \frac{ds}{R_s} - \gamma \int_0^t R_s\, ds \tag{4}$$

where $a = |\vec{a}|$ and $(\tilde{B}_t, t \geq 0)$ is a 1-dimensional Brownian motion, starting from 0.

- The main result of this note is the following :

Proposition 1. *Define the probability $\mathbb{Q}_a^\gamma$ via :*

$$\mathbb{Q}_a^\gamma|_{\mathcal{F}_t} = \frac{X_{t\wedge T_o}}{a}\, e^{\gamma t}.\, \mathbb{P}_a^\gamma|_{\mathcal{F}_t} \tag{5}$$

Then, $\mathbb{Q}_a^\gamma$ is the law of the 3-dimensional radial Ornstein-Uhlenbeck process starting from a.

Proof. Due to Girsanov's theorem, and (2), (B_t) considered under $\mathbb{Q}_a^\gamma$ solves :

$$B_t = \tilde{B}_t + \int_0^{t\wedge T_0} \frac{\exp(\gamma s)}{X_s\, e^{\gamma s}}\, ds \tag{6}$$

Now, noting that $T_0 = \infty$, $\mathbb{Q}_a^\gamma$ a.s, we obtain that X_t under $\mathbb{Q}_a^\gamma$ satisfies (4), as a consequence of (2) and (6). □

- We now make a few comments :

1. Note that, for $\gamma = 0$, (5) is nothing else but Doob's h-transform relationship between Brownian motion killed when it reaches 0 and the 3-dimensional Bessel process.
2. We now write (5) in the equivalent form :

$$1_{\{t<T_0\}} \, . \, \mathbb{P}_a^\gamma|_{\mathcal{F}_t} = \frac{a}{X_t} \, e^{-\gamma t} . \, \mathbb{Q}_a^\gamma|_{\mathcal{F}_t} \tag{7}$$

   which is nothing else but the identity in law obtained by P. Salminen ([6], Theorem 1,(i)) between, on the left hand-side, the Ornstein-Uhlenbeck process with parameter γ, killed when it reaches 0, and on the right hand-side the 3-dimensional radial Ornstein-Uhlenbeck process killed exponentially at rate γ, and conditioned to hit 0, that is the density $(\frac{1}{X_t})$ corresponding to such a conditioning may be seen a posteriori from the formula :

$$\mathbb{Q}_a^\gamma(F_t \frac{1}{X_t}) = \mathbb{Q}_a^\gamma(\frac{1}{X_t}) \, \mathbb{P}_a^\gamma(F_t | t < T_0), \; \forall \, F_t \in \mathcal{F}_t \, , \tag{8}$$

   which is a consequence of(7)
   It also follows from (7) that $\left(\frac{1}{X_t} \exp\left(-\gamma t\right), t \geq 0\right)$ is a strictly local martingale with respect to $\mathbb{Q}_a^\gamma$, thus extending very simply the well-known result for the case $\gamma = 0$, when (X_t) is a 3-dimensional Bessel process.
3. Note that (7) leads us to the computation of the semi-group of the killed Ornstein-Uhlenbeck, via :

$$\mathbb{E}_a^\gamma[(\frac{e^{\gamma t}}{a}) \, f(X_t) \, 1_{\{t \wedge T_0\}}] = \mathbb{Q}_a^\gamma[f(X_t) \, \frac{1}{X_t}] \, , \tag{9}$$

   allowing us to recover the result in [7], p122, formula (53).
4. What about $\gamma < 0$? Formula (5) is still valid thanks to the same arguments; it is closely related to Theorem 1,(ii) of [6].
5. Related informations about (radial) Ornstein-Uhlenbeck processes, and their hitting times, may be found in [1], [3], [4], [5].

Acknowledgments : I am very grateful to P. Salminen for a number of discussions during the preparation of this note.

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